ORBIT THEORY FOR ACCELERATORS

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Abstract

All the devices in synchrotrons and storage rings are placed at fixed positions. Observations are also made at fixed positions. Thus, the orbit length is a natural independent variable, and (arrival) time and energy are canonical variables. This description is a commonplace for betatron oscillations, synchrotron oscillations in a static case, and for collective beam instabilities. However, the time is usually used for synchrotron oscillations in a changing magnetic fields. Also, betatron accelerations are sometimes ignored. We develop a symplectic theory for synchrotron oscillations which uses the orbit length as an independent variable and includes betatron acceleration. Since sychrotron oscillations are closly connected with transverse coordinates, we also study synchro-betatron coupling.

1 INTRODUCTION

We develop an orbit theory for circular accelerators using the orbit length s as an independent variable (s-description). Chao [1] stressed the difference between a snap-shot (a picture taken at a fixed time, t-description) , and observations at fixed places (s-description). He developed an orbit theory for collective effects from this point of view. The equation of betatron oscillations is described in this way, but synchrotron oscillations are usually studied by the t-description.

The t-desription has several defects. Firstly, it is difficult to describe localized natures of rf cavities, etc. We are forced to use a travelling-wave approximation. A standing-wave picture, which is more physical, predicts chaotic behaviors in synchrotron oscillations when the synchrotron tune is large [2]. Then, the concept of rf-buckets breaks down. Also, Piwinski [3] showed in a linear approximation that synchrotron tunes are different in the travellin-wave approximations and in the standing-wave treatment. For a high tune, synchrotron oscillations become unstable. Further, when we study synchro-betatron coupling, we are forced to use two independent variables. Also, the standing-wave picture is necessary to find a resonance condition $\nu_x = n + m\nu_s$,where ν_x and ν_s are betatron and synchrotron tunes, and n and m are arbitrary integers. In the travelling-wave picture, only $n = 0$ effects appear.

These are commonplace in static cases, but are also important in the case of changing magnetic fields. In this case, betatron acceleration driven by the changing magnetic fields is somtimes neglected, but this must be included. Bryant and Johnsen [4] analyzed this point in detail in the t-description.It is interesting to note that Veksler [5] and McMillan [6] used the s-description. We develop a symplectic theory for synchrotron oscillations and synchro-betatron coupling from the viewpoints described above.

2 CANONICAL VARIABLES AND EQUATIONS OF MOTION

In the s-description, the (arrival) time t and minus the energy $-E$ are canonical variables. We first make a canonical transformation from t to τ by a relation $t = t_0 + \tau$, where t_0 is the arrival time of the synchronous particle $t_0(s) = \int ds/v_0$, v_0 the velocity of the synchronous particle, and $\tau(s)$ is the time delay of an arbitrary particle. We put a subscript 0 to variables of the synchronous particle in this paper except for β and γ. Then we make the second canonical transformation from $(\tau, -E)$ to $(\tau, -\Delta E)$ by a relation $E = E_0 + \Delta E$, where ΔE is the energy error.

Though the equations of motion can be derived from a Hamiltonian, we can obtain them from physical considerations if we pay due attentions to canonical natures of the variables. We describe this simplified approach though the equations are checked by a Hamiltonian formalism. The energy equation is

$$\frac{d\Delta E}{ds} = eV\delta_p(s - s_0)\{\sin\phi - \sin\phi_0\} + e\dot{B}x, \qquad (1)$$

where $eV$is the peak energy gain by rf-cavities, ϕ is the rf phase, s_0 is the position of the rf cavity, δ_p is the periodic δ-function, $\dot{B}$ is the time derivative of a vertical magnetic induction, and x is the horizontal coordinate. In this paper, the dot means a partial or a total derivative with repect to time. The time equation is derived from simple geometrial considerations and, after a few steps, we obtain

$$\frac{d\tau}{ds} = \frac{1}{v_0}(\frac{x}{\rho} - \frac{1}{\beta^2\gamma^2}\frac{\Delta E}{E_0}) \qquad (2)$$

where only linear terms are kept. The pair (x, p_x) denotes canonical variables for the transverse motion.

Now, x is decomposed as

$$x = x_\beta + D(\frac{\Delta E}{\beta^2 E_0} - \frac{\Delta B}{B_0}) + x_{co}, \qquad (3)$$

where x_β is the coordinate of betatron oscillations, D is the dispersion function, ρ is the radius of curvature, β is the velocity Lorentz factor, ΔB is the field error and x_{co} denotes a closed orbit distortion driven by errors. Usually, only the $\Delta E/E_0$ term in Eq.(3) is kept for synchrotron oscillations, but the ΔB-term is also important for a symplectic description. Different particles pass through a fixed point s at different times and feel different magnetic field strength. Thus,

$$\Delta B(t_o + \tau) = \Delta B(t_0) + \dot{B}(t_0)\tau \qquad (4)$$

Since $\Delta B(t_0)$ term does not contain any canonical variable, it only affects a closed orbit, but the $\dot{B}$ term is important. If this term is neglected, the necessry condition for symplecticity

$$\frac{\partial \Delta E'}{\partial \Delta E} + \frac{\partial \tau'}{\partial \tau} = 0 \tag{5}$$

is not satisfied. In this paper, the primes denote differentiation with respect to s. This condition is also a sufficient condition for the Louville theorem. The x_β-term shows a synchro-betatron coupling. We neglect x_{co} because this does not oscillate and also it is small.

We now express Eq.(3) by a canonical transformation with a generating function that has old coordinates and new momenta.

$$F = F_1 + F_2 + F_3, \tag{6}$$

where

$$F_1 = \overline{p_\beta}\{x - D(\frac{\overline{\Delta E}}{\beta^2 E_0} - \frac{\dot{B}\tau}{B_0})\},$$

$$F_2 = xD' p_0(\frac{\overline{\Delta E}}{\beta^2 E_0} - \frac{\dot{B}\tau}{B_0}),$$

$$F_3 = -\overline{\Delta E}\tau - \frac{1}{2} DD' p_0(\frac{\overline{\Delta E}}{\beta^2 E_0} - \frac{\dot{B}\tau}{B_0})^2.$$

Here, $\overline{p_\beta}$ is a canonical momentum conjugate to $\overline{x_\beta}$ and p_0 is the kinetic momentum. This generating function was first obtained by Morton and Chao[7] with an approximation $D' = \dot{B} = 0$. Corsten and Hagedoorn[8] included the D'-term and the $\dot{B}$-term is now included.

The relations between the old and new variables are given as

$$x = \overline{x_\beta} + D\{\frac{\overline{\Delta E}}{\beta^2 E_0} - \frac{\dot{B}}{B_0}(\bar{\tau} + \tau_\beta)\} \tag{7}$$

$$p_x = \overline{p_\beta} + D' p_0\{\frac{\overline{\Delta E}}{\beta^2 E_0} - \frac{\dot{B}}{B_0}(\bar{\tau} + \tau_\beta)\} \tag{8}$$

$$\Delta E = \overline{\Delta E} + \frac{\dot{B}}{B_0}\tau_\beta \beta^2 E_0 \tag{9}$$

$$\tau = \bar{\tau} + \tau_\beta \tag{10}$$

where

$$\tau_\beta = \frac{p_0 D' \bar{x}_\beta - \bar{p}_\beta D}{\beta^2 E_0} \tag{11}$$

The bars indicate the new canonical variables.

The meaning of τ_β can be seen by

$$\frac{d\tau_\beta}{ds} = \frac{\overline{x_\beta}}{\rho v_0} \tag{12}$$

where we neglected the adiabatic change of the parameters, and we used the equation of betatron oscillations and the defining equation for D. Thus, τ_β shows the time delay due to betatron oscillations. This relation was first found by Piwinski and Wrulich[9] by a heuristic manner. This quantity is also known as CP (Central Position) phase in the theory of cyclotrons. (See the references cited in [8].)

In the present case, we can include not only free betatron oscillations, but also synchro-betatron coupling. Also, we see that the $e\dot{B}\overline{x_\beta}$ term is cancelled when we use $\overline{\Delta E}$ instead of ΔE. Thus, the betatron acceleration by betatron oscillations is cancelled out: the betatron oscillations affect only the arrival time in synchrotron oscillations.

Inserting Eqs.(7) to (10) into Eqs.(1) and(2), we obtain the energy and the time equations expressed by the new canonical variables. From now on, we omit the bars from the new variables for the sake of simplicity. Neglecting the $\dot{B}^2$ term, we obtain

$$\frac{d\Delta E}{ds} = e\dot{B}D\frac{\Delta E}{\beta^2 E_0} + eV\delta_p(s - s_0)\{\sin\phi - \sin\phi_0\} \tag{13}$$

$$\frac{d\tau}{ds} = \frac{1}{v_0}\{(\frac{D}{\rho} - \frac{1}{\gamma^2})\frac{\Delta E}{\beta^2 E_0} - \frac{D\dot{B}}{\rho B}(\bar{\tau} + \tau_\beta)\} \tag{14}$$

We note that the coordinates of betatron oscillations x_β is cancelled out and only τ_β term remains. Together with the corresponding equations for x_β and p_β, we obtain the equations of motion for synchro-betatron coupling. For the static case in the s-description and in the standing-wave picture, the equations are given in [10] though several errors are present in this paper. There it is described that the standing -wave pictue (δ_p-function) is important to derive a resonace condition $\nu_x = n + m\nu_s$.

We now make a brief comment on rf phase angle $\phi = \phi_0 + \Delta\phi$. In the standing-wave picture, the particles feel an rf-field only at the position of rf-cavities. So, it is natural to put $\phi_0 = \omega_{rf}(t_0)t_0$. Also, we put $\Delta\phi = \omega_{rf}(t_0)\tau$ to the first order in τ. Such equations are described in a textbook by Livingston and Blewett[11] though in the t-description.

Now, we neglect the synchro-betatron coupling and put $\tau_\beta = 0$. We study pure synchrotron oscillations. We use a travelling-wave approximation here: The δ_p-finction is expanded into a Fourier series and we keep only one harmonic term with a harmonic nuber h. The phase angle φ in this case is redefined as $\varphi = \phi - h\theta$, where $\theta = s/R$ and R is the average radius. We further make a one-turn average of the quantity $< D/\rho >= \alpha$, where α is the momentum compaction factor. Then, we obtain

$$\frac{d\Delta E}{d\theta} = \frac{eV}{2\pi}\{\sin\varphi - \sin\varphi_0\} + \frac{\alpha\dot{B}}{\omega_0 B}\Delta E \tag{15}$$

$$\frac{d\tau}{d\theta} = \frac{1}{\omega_0}\{\eta\frac{\Delta E}{\beta^2 E_0} - \alpha\frac{\dot{B}}{B_0}\tau\} \tag{16}$$

where $\eta = \alpha - 1/\gamma^2$.

Combining Eqs. (15) and (16), we obtain an equation for τ

$$\frac{d}{d\theta}(\frac{\omega_0\beta^2 E_0}{\eta}\frac{d\tau}{d\theta}) = \frac{eV}{2\pi}(\sin\varphi - \sin\varphi_0) \tag{17}$$

where the second-order term $\dot{B}^2$ is consistantly omitted. We note that the $\dot{B}$ terms appear only in the second or higher order terms. This suggests that we can obtain a correct equation even in the absence of betatron accelerations, as analized by Bryant and Johnsen.

The arrival time at a fixed point has a strict physical significance, but people usually use the rf-phase. The canonical variable conjugate to $\Delta\phi$ is $W(=-\Delta E/\omega_{rf})$. Inserting these variables into Eqs.(15) and (16), combining the two as before, and neglecting the second and higher order terms in the adiabaticaly-changing variables, we obtain after several steps

$$\frac{d}{d\theta}(\frac{\beta^2 E_0}{h\eta\omega_0}\frac{d\Delta\varphi}{d\theta}) = \frac{eV}{2\pi\omega_0}\{\sin(\varphi_0+\Delta\varphi)-\sin\varphi_0\} \quad (18)$$

If we put $\alpha = 1$ (pure bending field) and $h = 1$,this equation reduces to McMillan's one. Also, if we put $d\theta = \omega_0 dt$, Eq.(18) reduces to the one given by Courant and Snyder.

3 DISCUSSIONS AND CONCLUSIONS

We developed an orbit theory for synchrotron oscillations and synchro-betatron coupling. We stressed the importance of a standing-wave picture. The travelling-wave concepts such as the rf-buckets are approximations. A smooth elliptical phase space trajectory is also an approximation. It consists of a drift (with a small betatron acceleration) and a sudden jump in energy by rf-cavities. The trajectory is polygonal. The standing-wave treatment has revealed the instability and also a chaotic behavior for high synchrotron tunes. On the other hand, the travelling-wave approximation is necessary for analytic works. Even in this case, the orbit theory is not more complicated than the t-description.

More details including a Hamiltonian formalism will be described elsewhere.

4 REFERENCES

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